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CALL ROUTING AND THE RATCATCHER

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Suppose we expect there to be $p(ab)$ phone calls between locations a and b, for all choices of a, b from some set L of locations. We wish to design a network to optimally handle these calls. More precisely, a "routing tree" is a tree T with set of leaves L, in which every other vertex has valency 3. It has "congestion" $<k$ if for every edge e of T, there are fewer than k calls which will be routed along e, that is, between locations a, b in different components of $T\backslash e$. Deciding if there is a routing tree with congestion $<k$ is NP-hard, but if the pairs ab with $p(ab)>0$ form the edges of a planar graph G, there is an efficient, strongly polynomial algorithm.

This is because the problem is equivalent to deciding if a ratcatcher can corner a rat loose in the walls of a house with floor plan G, where $p(ab)$ is a thickness of the wall ab. The ratcatcher carries a noisemaker of power k, and the rat will not move through any wall in which the noise level is too high (determined by the total thickness of the intervening walls between this one and the noisemaker).

It follows that branch-width is polynomially computable for planar graphs — that too is NP-hard for general graphs.

1. Introduction

Let G be a graph (all graphs in this paper are finite, and may have loops or multiple edges) and for every edge e of G let $p(e)\geq 0$ be an integer. A *routing tree* is a tree T with $V(G)\subseteq V(T)$, such that every $v\in V(G)$ has valency 1 in T and every $v\in V(T)-V(G)$ has valency 3 in T. ($V(G)$ denotes the vertex set of G.) If $k\geq 0$ is an integer, we say that T has *congestion* $<k$ if $\sum p(e)<k$ for every $f\in E(T)$, where the sum is taken over all $e\in E(G)$ with ends in different components of $T\backslash f$. ($E(G)$ denotes the edge set of G, and we use $\backslash$ for deletion.) As explained in the abstract, the problem of deciding if there is a routing tree of congestion $<k$ is relevant to telephone network design.

Our main result is that there is a strongly polynomial algorithm with running time $O(m^2)$, which, with input G, p and k as above with G planar and $|V(G)|+|E(G)|=m$, decides if there is a routing tree with congestion $<k$. We shall also show that for general graphs G the problem is NP-hard, even if $p(e)=1$ for all edges e.

It is convenient to work with "carvings" rather than with routing trees. These are related objects, and are defined as follows. Let V be a finite set with $|V|\geq 2$. Two subsets $A,B\subseteq V$ *cross* if $A\cap B$, $A-B$, $B-A$, $V-(A\cup B)$ are all non-empty. A *carving* in V is a set $\mathscr{C}$ of subsets of V such that

(i) $\emptyset$, $V\notin\mathscr{C}$

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(ii) no two members of $\mathscr{C}$ cross, and

(iii) $\mathscr{C}$ is maximal subject to (i) and (ii).

It is sometimes helpful to view a carving as arising from a tree, as follows. (The *leaves* of a tree are its vertices of valency 1.)

(1.1) *Let V be a finite set with $|V|\geq 2$, let T be a tree in which every vertex has valency 1 or 3, and let τ be a bijection from V onto the set of leaves of T. For each edge e of T let $T_1(e)$, $T_2(e)$ be the two components of $T\backslash e$; and let*

$$\mathscr{C} = \{\{v \in V \,:\, \tau(v) \in V(T_i(e))\} \,:\, e \in E(T), \quad i = 1, 2\}.$$

Then $\mathscr{C}$ is a carving in V. Conversely, every carving in V arises from some tree T and bijection τ in this way.

We omit the proof, which is easy.

Now let G be a graph. For $A \subseteq V(G)$, we denote by $\delta(A)$ or $\delta_G(A)$ the set of all edges with an end in A and an end in $V(G)-A$. For each $e \in E(G)$, let $p(e) \geq 0$ be an integer. For $X \subseteq E(G)$ we denote $\sum_{e\in X} p(e)$ by $p(X)$, and if $|V(G)| \geq 2$ we define the *p-carving-width* of G to be the minimum, over all carvings $\mathscr{C}$ in $V(G)$, of the maximum, over all $A \in \mathscr{C}$, of $p(\delta(A))$. It is easy to see, via (1.1), that G has p-carving-width $<k$ if and only if G has a routing tree of congestion $<k$ and so our basic problem may be reformulated in terms of carvings. The *carving-width* of G is the p-carving-width of G where $p(e)=1$ for every edge e. It is easy to see that in general, the p-carving-width of G equals the carving-width of the graph obtained from G by replacing each edge e by $p(e)$ parallel edges; but we do not use this reduction at this stage because we wish to design an algorithm which is strongly polynomial.

Again, let G be a graph. For $A \subseteq E(G)$ we denote by $\partial(A)$ or $\partial_G(A)$ the set of all $v \in V(G)$ incident with an edge in A and with an edge in $E(G)-A$. The *branch-width* of G is the minimum, over all carvings $\mathscr{C}$ in $E(G)$, of the maximum, over all $A \in \mathscr{C}$, of $|\partial(A)|$ (or zero, if $|E(G)| \leq 1$). Branch-width has been investigated in other papers, particularly [4], and is closely connected with "tree-width". We shall show that one can compute the branch-width of a planar graph in polynomial time, but computing it for general graphs is NP-hard.

Our method for branch-width is simply to prove that for a connected planar graph G, its branch-width is half the carving-width of a derived planar graph called the *medial graph* of G; and so we can use our carving-width algorithm. On the other hand, our method for computing the carving-width of a planar graph is quite indirect. We do not search for a low width carving, but for an obstruction to its existence called an *antipodality*. (Antipodalities are escape strategies for the rat, while low width carvings are search strategies for the ratcatcher, in a ratcatching game played on the graph which we discuss in section 3.) This is easy to search for; but proving that G has carving-width $<k$ if and only if G has no antipodality of "range" k is difficult, and uses some hard theorems from the "Graph Minors" series of papers of Robertson and Seymour. Fortunately, this means that the algorithm for computing carving-width is easy; it is only the proof of its correctness that is difficult.

2. Antipodalities

A *walk* in a graph G is a sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$, where $v_0, v_1, \ldots, v_k \in V(G)$, $e_1, \ldots, e_k \in E(G)$, and $\{v_{i-1}, v_i\}$ is the set of ends of $e_i (1 \le i \le k)$. It is *closed* if $v_0 = v_k$. Let Σ be a sphere, and let G be a graph drawn in Σ. We denote the set of regions of the drawing by $R(G)$. (Each region is an open set.) An edge e is *incident* with a region r if $e \subseteq \overline{r}$ (for $X \subseteq \Sigma$, $\overline{X}$ denotes the closure of X). Now let G be non-null and connected, and let G^* be a dual graph also drawn in Σ, in the usual sense of geometric duality. For each $v \in V(G)$ there is a unique $r \in R(G^*)$ with $v \in r$, and we define $v^* = r$. Similarly, for $r \in R(G)$, $r^* \in V(G^*)$ is the unique vertex of G^* in r; and for $e \in E(G)$, e^* is the unique edge of G^* crossing e.

Let $p: E(G) \to \mathbb{Z}_+$ (the set of non-negative integers). A walk

$$v_0, e_1, v_1, e_2, \ldots, e_k, v_k$$

in G^* has *p-length* $p(f_1) + \ldots + p(f_k)$, where $e_i = f_i^*$ $(1 \le i \le k)$. An *antipodality* in G of *p-range* $\ge k$ is a function α with domain $E(G) \cup R(G)$, such that for all $e \in E(G)$, $\alpha(e)$ is a non-null subgraph of G, and for all $r \in R(G)$, $\alpha(r)$ is a non-empty subset of $V(G)$, satisfying:

(A1) If $e \in E(G)$ then no end of e belongs to $V(\alpha(e))$,

(A2) If $e \in E(G)$, $r \in R(G)$, and e is incident with r, then $\alpha(r) \subseteq V(\alpha(e))$, and every component of $\alpha(e)$ has a vertex in $\alpha(r)$,

(A3) If $e \in E(G)$ and $f \in E(\alpha(e))$ then every closed walk of G^* using e^* and f^* has p-length $\ge k$.

For example, let G be the graph of the octahedron (that is, K_6 with a perfect matching deleted), drawn in the plane. For each region r, let $\alpha(r)$ be the three vertices not incident with r, and for each edge e, let $\alpha(e)$ be the graph obtained from G by deleting the ends of e. Then α is an antipodality of p-range 6 in G (where $p \equiv 1$).

The main result of this paper is that a connected planar graph with ≥ 2 vertices has p-carving-width $\ge k$ if and only if either it has an antipodality of p-range $\ge k$ or $p(\delta(v)) \ge k$ for some $v \in V(G)$. (We write $\delta(v)$ for $\delta(\{v\})$.) In other words, the minimum k such that there is a carving $\mathscr{C}$ of G with $p(\delta(A)) \ge k$ for all $A \in \mathscr{C}$ equals the maximum k such that G has either an antipodality of p-range $\ge k$ or a vertex v with $p(\delta(v)) \ge k$. This is quite a difficult theorem, but it provides a simple algorithm to determine p-carving-width, because one can test easily if G has an antipodality of p-range $\ge k$. Explaining how to do so is the objective of this section.

Let N be a simple graph, with vertex set I say. Let M be a simple graph, and let $(X_i : i \in I)$ be a partition of $V(M)$, such that if $u \in X_i$, $v \in X_j$ are adjacent in M then $i \ne j$ and i, j are adjacent in N. A set $R \subseteq V(M)$ is *round* if for all $i, j \in I$ adjacent in N, every vertex in $X_i \cap R$ is adjacent to some vertex in $X_j \cap R$. We need to be able to test if there exists a non-null round set. To do so, we use the following lemma.

(2.1) *Let $R \subseteq V(M)$ be round, and let $R \subseteq S \subseteq V(M)$. Suppose that $u \in S \cap X_i$ has no neighbour in $S \cap X_j$, and yet i and j are adjacent in N. Then $R \subseteq S - \{u\}$.*

Proof. If $u \in R$ then since R is round, u has a neighbour in $R \cap X_j \subseteq S \cap X_j$, a contradiction. Hence $u \notin R$. ∎

Because of (2.1) there is a greedy algorithm to test if there exists a non-null round set. Initially we set $S = V(H)$, and in general we will have some subset $S \subseteq V(M)$ which is guaranteed to include every round set. We check if for some i, j adjacent in N, some $u \in S \cap X_i$ has no neighbours in $S \cap X_j$. If so, we replace S by $S - \{u\}$ and repeat; this is still guaranteed to include all round sets, by (2.1). If there is no such u then S itself is round. If $S \neq \emptyset$ then there is a non-empty round set, and if $S = \emptyset$ there isn't.

We see that the algorithm finds a round set which includes all other round sets. That such a set always exists is clear, because the union of any two round sets is round, and so there is a unique maximal round set.

This process is obviously polynomially-bounded, but it is important for us to do it very quickly, and so we give a more careful description. Let H be the bipartite graph with vertex set $V(M) \cup I$, in which for all i, $j \in I$ adjacent in N, every $v \in X_i$ is adjacent in H to j. We refer to the edges of H as pairs vj.

(2.2) *Algorithm.*
Input: Graphs M, N, H and a partition $(X_i : i \in I)$ as above; and for all $vj \in E(H)$, the number $d(vj)$ of vertices in X_j adjacent to v in M.
Output: The maximal round subset of $V(M)$.
Running time: $O(|E(H)| + |V(M)| + |E(M)|)$.
Description:

Step 1. *Construct a stack L of all vertices $v \in V(M)$ such that $d(vj) = 0$ for some $j \in I$, without repetition. In G, label every vertex occurring in L "observed", and the remainder "unobserved".*

Since we are given the numbers $d(vj)$ we can construct L and the labelling in time $O(|E(H)|)$.

Step 2. *Set $S = V(M)$. Set $c(vj) = d(vj)$ for all $vj \in E(H)$.*

Now we begin a recursion. At the beginning of the ith iteration, S will have cardinality $V(M) - i + 1$, and will include the maximal round subset of $V(M)$. For $v \in S$ and $vj \in E(H)$, the number $c(vj)$ will be the number of neighbours of v in $S \cap X_j$, and L will consists precisely of all $v \in S$ such that $c(vj) = 0$ for some $vj \in E(H)$, without repetition. Every vertex of S will be labelled "observed" or "unobserved", and the former will be those which occur in L. The purpose of the labels is to avoid adding a vertex to L more than once; once a vertex is added to L we label it "observed", to warn ourselves not to add it again. The ith iteration proceeds as follows.

Step 3. *If L is empty, output S and stop.*

This works because if L is empty then S is round, and hence is the maximal round subset.

Step 4. *If L is non-empty, select $u \in L$. Let $u \in X_i$ where $i \in I$. Find the set $Y \subseteq S$ of all neighbours of u in S. Remove u from L; remove u from S; for each $v \in Y$*

labelled "unobserved", if $c(vi)=1$, change the label of v to "observed" and add v to L; and for each $v \in Y$, reduce $c(vi)$ by 1. Return for the next iteration.

It is easy to check that this algorithm outputs what it claims. For its running time, we observe that there are at most $|V(M)|+1$ iterations, since in each iteration $|S|$ is reduced by 1. The time spent in the ith iteration is $\leq k_1+k_2 d(u)$, where u is the vertex chosen in step 4, $d(u)$ is the valency of u in M, and k_1, k_2 are constants. Thus the total time spent in step 4 is

$$\leq k_1|V(M)| + 2k_2|E(M)|$$

and hence the algorithm has running time $O(|E(H)|+|V(M)|+|E(M)|)$. ∎

Let us see how to use (2.2) to test if a connected planar graph has an antipodality of p-range $\geq k$. We need the following.

(2.3) *Let G be a non-null connected planar graph with a dual graph G^*, let $p: E(G) \to \mathbb{Z}_+$, and let $k \geq 0$ be an integer. For each $e \in E(G)$ let $\phi(e)$ be the subgraph of G consisting of all vertices of G except the ends of e, and all edges f such that f is incident with no end of e and no closed walk of G^* of p-length $<k$ contains both e^* and f^*. If there is an antipodality in G of p-range $\geq k$, then there is one, α say, such that $\alpha(e)$ is a union of components of $\phi(e)$ for each $e \in E(G)$.*

Proof. Let β be an antipodality of p-range $\geq k$. Then for all $e \in E(G)$, $\beta(e)$ is a subgraph of $\phi(e)$, by (A1) and (A3). Let $\alpha(e)$ be the union of all components of $\phi(e)$ which intersect $\beta(e)$, for each $e \in E(G)$, and for each $r \in R(G)$ let $\alpha(r)=\beta(r)$. Clearly α satisfies (A1) and (A3). For (A2), let $e \in E(G)$ be incident with $r \in R(G)$. Then $\alpha(r)=\beta(r) \subseteq V(\beta(e)) \subseteq V(\alpha(e))$; and every component of $\alpha(e)$ includes a component of $\beta(e)$ and hence intersects $\beta(r)=\alpha(r)$. Thus, α is an antipodality of p-range $\geq k$, as required. ∎

We use (2.2), (2.3) for the following.

(2.4) *Algorithm.*
Input: A non-null connected planar graph G, a dual graph G^*, a function $p: E(G) \to \mathbb{Z}_+$, and an integer $k \geq 0$.
Output: Decides whether there is an antipodality in G of p-range $\geq k$.
Running time: $O(m^2)$ where $m=|V(G)|+|E(G)|$, if arithmetic operations can be performed in unit time.
Description:

Step 1. *For all u, $v \in V(G^*)$, compute $d^*(u,v)$, the minimum p-length of all paths G^* between u and v.*

This can be done using the algorithm of [2].

If e, $f \in E(G)$ are distinct and e^*, f^* have ends u_1, u_2 and v_1, v_2 say in G^*, then there is a closed walk of G^* of p-length $<k$ using e^* and f^* if and only if either $d^*(u_1,v_1)+d^*(u_2,v_2) < k-p(e)-p(f)$ or $d^*(u_1,v_2)+d^*(u_2,v_1) < k-p(e)-p(f)$. Thus we can use the numbers $d^*(u,v)$ computed in step 1 to perform

Step 2. *Compute the graph $\phi(e)$ of (2.3), for each $e \in E(G)$. Compute the set C_e of components of $\phi(e)$.*

For each $e \in E(G)$, let X_e be the set of all pairs $\{(e,C): C \in C_e\}$. For each $r \in R(G)$, let X_r be the set of all pairs $\{(r,v): v \in V(G)\}$. Let $I=E(G) \cup R(G)$.

Step 3. *Construct the graph M with vertex set $\bigcup(X_i : i \in I)$, in which $(e,C) \in X_e$ is adjacent to (r,v) in X_r if $e \in E(G)$, $r \in R(G)$, e is incident with r, and $v \in V(C)$; and construct the partition $(X_i : i \in I)$ of $V(M)$.*

Step 4. *Construct the graph N with vertex set $E(G) \cup R(G) = I$, in which $e \in E(G)$ is adjacent to $r \in R(G)$ if e is incident with r in G.*

Step 5. *Construct the graph H with vertex set $V(M) \cup V(N)$, in which if $e \in E(G)$ and $r \in R(G)$ are adjacent in N then e is adjacent in H to every vertex in X_r and r is adjacent in H to every vertex in X_e.*

Step 6. *For each edge vj of H, set*

$$d(vj) = \begin{cases} |V(C)| & \text{if } j \in R(G) \text{ and } v = (e,C) \text{ for some } e \in E(G) \text{ and } C \in C_e \\ 1 & \text{if } j \in E(G) \text{ and } v = (r,u) \text{ for some } r \in R(G) \text{ and } u \in V(G), \\ & \quad \text{where } u \text{ is not an end of } j \\ 0 & \text{if } j \in E(G) \text{ and } v = (r,u) \text{ for some } r \in R(G) \text{ and } u \in V(G), \\ & \quad \text{where } u \text{ is an end of } j. \end{cases}$$

Step 7. *Use (2.2) to decide if $V(M)$ has a non-empty round subset. Output "yes" or "no" accordingly. Stop.*

Step 7 is permissible because, by our definition of $d(vj)$ in step 7, M, N, H, the numbers $d(vj)$ and the sets $X_i (i \in I)$ satisfy the hypothesis of (2.2). The algorithm works correctly because $V(M)$ has a non-empty round subset if and only if G has an antipodality of p-range $\geq k$, as we show as follows. Let $R \subseteq V(M)$; then for $e \in E(G)$ we may correspondingly define $\alpha(e)$ to be the union of all $C \in C_e$ such that $(e,C) \in R$, and for $r \in R(G)$ define $\alpha(r)$ to be

$$\{v \in V(G) : (r,v) \in R\}.$$

For any choice of R the function α satisfies (A1) and (A3); it satisfies (A2) if and only if R is round; and the selected $\alpha(e)$, $\alpha(r)$ are non-null (one is non-null if and only if they all are, by (A2)) if and only if $R \neq \emptyset$. Since by (2.3) if there is an antipodality of p-range $\geq k$ then there is one arising in this way from some choice of R, we deduce that $V(M)$ has a non-empty round subset if and only if G has an antipodality of p-range $\geq k$. Thus the algorithm works correctly.

To estimate running time, let $m = |V(G)| + |E(G)|$. Then $|V(G^*| \leq m$, by Euler's formula. Steps 1–6 each take time $O(m^2)$. Step 7 takes time $O(|E(H)| + |V(M)| + |E(M)|) \leq O(m^2)$. Thus, the whole algorithm takes time $\leq O(m^2)$. ∎

3. Ratcatching

An antipodality is nothing more than an escape strategy in a certain searching game, and in this section we discuss the game. This is not relevant to the algorithm, or to the proof of its correctness, and is included for its own interest.

Let G be a connected planar graph drawn in a sphere Σ, with dual graph G^*, and let $p : E(G) \to \mathbb{Z}_+$ be a function. We regard G as the floor-plan of a one-storey house; its region are rooms, its edges are walls, and its vertices are corners. For each $e \in E(G)$, the wall e has thickness $p(e)$. Here is a full-knowledge game for two players (ratcatcher and rat). An integer $k \geq 0$ is fixed, the ratcatcher selects a room, and the rat select a corner, and the first move begin. The ratcatcher moves first, and the players move in turn, unless the ratcatcher wins when the game stops. When it is the ratcatcher's turn, if he is currently in a room, he moves to an incident wall; or if he is currently in a wall (the walls have doorways) he moves to the room incident with the wall on the other side from which he entered the wall. (In other words, once the ratcatcher moves to a doorway out of a room, he has to go through it in his next turn to the next room; he cannot move to the doorway and then change his mind and return to the first room.) When it is the rat's turn, and it is currently in a corner v, it moves to a corner u (possibly $u=v$) such that there is a path P of G with ends u, v, and every edge of P is currently "quiet". We say an edge e of G is *quiet* if there is no closed walk in G^* of p-length $<k$ using e^* and r^* if the ratcatcher is in room r, or using e^* and f^* if the ratcatcher is in wall f. (In terms of ratcatching, the rat runs from corner to corner inside the walls; but the ratcatcher has a noise-maker, and the rat only moves along walls in which the noise level is acceptable.) The ratcatcher wins if the rat is in a corner v with $p(\delta(v))<k$, and the ratcatcher is in a room incident with v. The rat's objective is to stop the ratcatcher winning.

We claim that:

(3.1) *If $|V(G)| \geq 2$, then the ratcatcher has a winning strategy if and only if $p(\delta(v))<k$ for every $v \in V(G)$ and there is no antipodality of p-range $\geq k$.*

Proof. Certainly if $p(\delta(v)) \geq k$ for some v, the rat can survive by remaining at v, and so we assume that there is no such v. If α is an antipodality of p-range $\geq k$, the rat can survive by obeying the following rules:

(1) Initially, if the ratcatcher selects room r, the rat selects a corner in $\alpha(r)$

(2) In general, if the ratcatcher moves from a room r to a doorway in a wall e, incident with rooms r and s, the rat moves into $\alpha(s)$.

(3) If the ratcatcher moves from a doorway in a wall to a room, the rat remains still.

It is possible to obey these rules, because in (2) the rat is currently in $\alpha(r)$ and hence in some component C of $\alpha(e)$; and every edge of C is quiet, and some vertex of C is in $\alpha(s)$. Moreover, it follows from (A1), (A2) that the ratcatcher does not win if the rat obeys these rules.

Conversely, suppose the rat has an escape strategy. We may assume that the rat only moves when the ratcatcher is in a doorway; for when the ratcatcher is in a doorway and is about to move into room s, the rat knows which room the ratcatcher is about to enter, and every edge which will be quiet when the ratcatcher is in s is quiet already, and so there is nothing to be gained by waiting to move. For each room r, define $\alpha(r)$ to be the set of all corners v such that the rat can guarantee to survive if the game starts with ratcatcher in r and rat in v, with ratcatcher to move. For each wall e, incident with rooms r, s, let H be the subgraph of G with vertex set $V(G)$ and edges the currently quiet edges of G, and let $\alpha(e)$ be the union of those components of H which meet $\alpha(r)$ and $\alpha(s)$. We claim that α is an antipodality of p-range $\geq k$. Now (A1) holds since $p(\delta(v))<k$ for all $v \in V(G)$, and (A3) from the definition of H. To see (A2), let $e \in E(G)$ be incident with $r, s \in$

$R(G)$. Certainly every component of $\alpha(e)$ meets $\alpha(r)$ from the definition of $\alpha(e)$. Let $v \in \alpha(r)$, and suppose that the rat is in v, and the ratcatcher moves from r to e. By definition of $\alpha(r)$, the rat can still guarantee to survive, and so there is a path of H (defined as before) from v to a vertex in $\alpha(s)$. Hence $v \in V(\alpha(e))$, and so (A2) holds. Finally, $\alpha(r) \neq \emptyset$ for each $r \in R(G)$ since otherwise the ratcatcher wins by moving to r; and consequently each $\alpha(e)$ is non-null, by (A2). ∎

4. Slopes

The next three sections are devoted to proving the following. ($\mathbb{N}$ denotes the set of positive integers.)

(4.1) *Let G be a connected planar graph with $|V(G)| \geq 2$, let $p: E(G) \to \mathbb{N}$, and let $k \geq 0$ be an integer. Then G has p-carving-width $\geq k$ if and only if either $p(\delta(v)) \geq k$ for some vertex v, or G has an antipodality of p-range $\geq k$.*

In this section we prove "only if". The proof is in several steps. We shall need the following theorem; it is [4, theorems (3.5) and (3.6)] in the case when $\mathcal{A} = \{\{v\} : v \in V\}$, rephrased in terms of carvings via (1.1).

(4.2) *Let V be a finite set with $|V| \geq 2$, and for each $X \subseteq V$ let $\kappa(X)$ be an integer, such that*
- (i) *$\kappa(X) = \kappa(V - X)$ for all $X \subseteq V$*
- (ii) *$\kappa(X \cup Y) + \kappa(X \cap Y) \leq \kappa(X) + \kappa(Y)$ for all $X, Y \subseteq V$*
- (iii) *$\kappa(X) \leq 0$ for all $X \subseteq V$ with $|X| = 1$.*

Then exactly one of the following holds:
- (a) *there is a carving $\mathscr{C}$ in V such that $\kappa(X) \leq 0$ for all $X \in \mathscr{C}$*
- (b) *there is a set of $\mathscr{B}$ of subsets of V, such that*
  - (i) *for $X \subseteq V$, $\mathscr{B}$ contains one of X, $V - X$ if and only if $\kappa(X) \leq 0$*
  - (ii) *if $X, Y, Z \in \mathscr{B}$ then $X \cup Y \cup Z \neq V$*
  - (iii) *$X \in \mathscr{B}$ for all $X \subseteq V$ with $|X| = 1$.*

Let G be a graph, and $p: E(G) \to \mathbb{N}$ a function. Let us say a *tilt* in G of *p-order* k is a collection $\mathscr{B}$ of subsets of $V(G)$ such that
- (B1) for $X \subseteq V(G)$, $\mathscr{B}$ contains one of X, $V(G) - X$ if and only if $p(\delta(X)) < k$
- (B2) if $X, Y, Z \in \mathscr{B}$ then $X \cup Y \cup Z \neq V(G)$
- (B3) $X \in \mathscr{B}$ for all $X \subseteq V(G)$ with $|X| = 1$.

From (4.2), we deduce

(4.3) *Let G be a graph with $|V(G) \geq 2$, let $p: E(G) \to \mathbb{N}$, and let $k \geq 1$ be an integer, such that $p(\delta(v)) < k$ for all $v \in V(G)$. Then G has p-carving-width $\geq k$ if and only if G has a tilt of p-order k.*

Proof. For each $X \subseteq V(G)$, let $\kappa(X) = p(\delta(X) - k + 1$. Thus $\kappa(X) \leq 0$ if and only if $p(\delta(X)) < k$. Then κ satisfies the hypothesis (i), (ii), (iii) of (4.2). By (4.2), exactly one of (4.2)(a), (4.2)(b) hold. But (4.2)(a) holds if and only if G has p-carving-width $< k$, and (4.2)(b) holds if and only if G has a tilt of p-order k. ∎

Now let G be a graph drawn in a sphere Σ, and let $k \geq 1$ be an integer. A *slope* in G of *order* $k/2$ is a function *ins* which assigns to every circuit C of G of length

$< k$ a closed disc $ins(C) \subseteq \Sigma$ that is one of the two closed discs bounded by C in the drawing, such that

(S1) if C, C' are circuits of length $< k$, and C is drawn within $ins(C')$ then $ins(C) \subseteq ins(C')$

(S2) If P_1, P_2, P_3 are three paths of G joining the same pair u, v of distinct vertices but otherwise disjoint, and the three circuits $P_1 \cup P_2$, $P_2 \cup P_3$, $P_3 \cup P_1$ all have length $< k$, then

$$ins(P_1 \cup P_2) \cup ins(P_2 \cup P_3) \cup ins(P_3 \cup P_1) \neq \Sigma.$$

A slope is *uniform* if for every $r \in R(G)$ there is a circuit C of G with length $< k$ such that $r \subseteq ins(C)$. If $X \subseteq V(G)$ we denote by $G \mid X$ the subgraph of G induced by X, that is, $G \backslash (V(G) - X)$. If G is connected, and X, $Y \subseteq V(G)$ are disjoint with union $V(G)$, and $G \mid X$, $G \mid Y$ are both non-null and connected, we call $\delta(X)$ a *bond* of G.

Let G be a graph drawn in Σ and let e be an edge of G, with ends u, v. If we select t points of Σ from the open line segment of the drawing representing e, and declare them to be vertices, we obtain a drawing of a new graph in which the edge e has been replaced by a $(t+1)$-edge path. This process is called *subdividing e t times*.

(4.4) *Let G be a non-null connected graph drawn in a sphere Σ, and let G^* be a dual graph. Let $p: E(G) \to \mathbb{N}$, and let G' be obtained from G^* by subdividing e^* $p(e) - 1$ times, for each $e \in E(G)$. Let $k \geq 1$ be an integer, such that $p(\delta(v)) < k$ for all $v \in V(G)$. If G has a tilt of p-order k, then G' has a uniform slope of order $k/2$.*

Proof. Let $\mathscr{B}$ be a tilt in G of p-order k. For each circuit C of G' of length $< k$, let Δ_1, Δ_2 be the two closed discs bounded by C in the drawing. Then $p(\delta(V(G) \cap \Delta_i)) = |E(C)| < k$ $(i = 1, 2)$, and so exactly one of $V(G) \cap \Delta_1$, $V(G) \cap \Delta_2$ belongs to $\mathscr{B}$, say $V(G) \cap \Delta_1$. We define $ins(C) = \Delta_1$. It is easy to see that ins is a slope in G' of order $k/2$, because of (B1), (B2). To see that ins is uniform we proceed as follows. Let $r \in R(G')$; then $r \in R(G^*)$, and $r = v^*$ for some $v \in V(G)$. Now $\{v\} \in \mathscr{B}$ (by(B3)) and $G \mid \{v\}$ is connected. Choose $X \in \mathscr{B}$ maximal such that $v \in X$ and $G \mid X$ is connected. Let $Y = V(G) - X$, and let $Y_1, \ldots, Y_t$ be the vertex sets of the components of $G \mid Y$. We shall show that $t = 1$. Now for $1 \leq i \leq t$, $G|X \cup Y_i$ is connected, because $G \mid X$ and $G \mid Y_i$ are both connected and $\delta(Y_i) \neq \emptyset$ (since G is connected), and $\delta(Y_i) \subseteq \delta(X)$ (since Y_i is the vertex set of a component of $G \mid Y$). From the maximality of X it follows that $X \cup Y_i \notin \mathscr{B}$. But $\delta(X \cup Y_i) \subseteq \delta(X)$, and so $p(\delta(X \cup Y_i)) < k$; and hence

$$Y - Y_i = V(G) - (X \cup Y_i) \in \mathscr{B},$$

from (B1). If $t \geq 2$ then $Y - Y_1$, $Y - Y_2$, $X \in \mathscr{B}$, and $(Y - Y_1) \cup (Y - Y_2) \cup X = V(G)$, contrary to (B3). Thus $t \leq 1$, and $t \neq 0$ by (B2). Hence $G \mid Y$ is non-null and connected, and so $\delta(X)$ is a bond of G. It follows that $\{e^* : e \in \delta(X)\}$ is the edge-set of a circuit C of G^*. Let C' be the corresponding circuit of G'; then $|E(C')| < k$, and $r \subseteq ins(C)$, since $X \in \mathscr{B}$. Thus ins is uniform, as required. ∎

We shall need the following theorem, which follows from [5, theorems (8.7) and (8.9)].

(4.5) *Let G be a graph drawn in a sphere Σ, let $k \geq 1$ be an integer, let ins be a slope in G of order $k/2$, and let $x \in \Sigma$. Let N_x be the set of all $y \in \Sigma$ such that there is a closed walk in G of length $<k$ capturing x and y. Then either $\Sigma - N_x$ is an open disc or $N_x = \emptyset$; and if ins is uniform then $N_x \neq \emptyset$.*

(A walk W *captures* $x \in \Sigma$ if either it passes through x, or there is a circuit C of length $<k$ every edge of which belongs to W, with $x \in ins(C)$.)

An antipodality α is *connected* if $\alpha(e)$ is connected for all edges e. From (4.5) we deduce the following.

(4.6) *Let G, G^*, G', p, k be as in (4.4). If G' has a uniform slope of order $k/2$ then G has a connected antipodality of p-range $\geq k$.*

Proof. For each $x \in \Sigma$ let N_x be as in (4.5) (with G replaced by G'). For $r \in R(G)$ let $\alpha(r) = \{v \in V(G) : v^* \subseteq \Sigma - N_r^*\}$. For $e \in E(G)$, we define $\alpha(e)$ as follows. Let $x(e)$ be the point of intersection of the edges e, e^* in Σ. Let $\alpha(e)$ be the subgraph of G consisting of all $v \in V(G)$ with $v^* \subseteq \Sigma - N_{x(e)}$ and all $f \in E(G)$ with $f^* \subseteq \Sigma - N_{x(e)}$. (This is a subgraph, for if $f \in E(G)$ is incident with $v \in V(G)$ and $f^* \subseteq \Sigma - N_{x(e)}$ then $v^* \subseteq \Sigma - N_{x(e)}$.) Since $N_{x(e)}$ is an open disc by (4.5), it follows that $\alpha(e)$ is a non-null connected subgraph of G. We claim that α is an antipodality of range $\geq k$. To see (A1), let $e \in E(G)$, and let v be an end of e in G. Since *ins* is uniform, there is a circuit C of G' of length $<k$ with $v^* \subseteq ins(C)$, and hence with $e^* \subseteq ins(C)$. Thus there is a closed walk of G' with length $<k$ capturing each point of v^* and capturing $x(e)$, and so $v^* \subseteq N_{x(e)}$. Hence $v \notin V(\alpha(e))$. This verifies (A1).

For (A2), let $e \in E(G)$ be incident with $r \in R(G)$. Then e^* is incident with r^* in G^*, and so $N_{x(e)} \subseteq N_r^*$, for any walk of G' capturing $x(e)$ also captures r^*. Thus $\Sigma - N_{r^*} \subseteq \Sigma - N_{x(e)}$, and so $\alpha(r) \subseteq V(\alpha(e))$. Since $\alpha(r)$ is non-null and $\alpha(e)$ is connected, this proves (A2).

For (A3), let $e \in E(G)$ and let $f \in E(\alpha(e))$. No closed walk of G' of length $<k$ captures both $x(e)$ and $x(f)$, and in particular no closed walk of G^* of p-length $<$ k contains both e^* and f^*. This proves (A3), as required. ∎

In summary, then, we have shown the following, by (4.3), (4.4) and (4.6).

(4.7) *Let G be a connected graph with $|V(G)| \geq 2$, drawn in a sphere Σ, let G^* be a dual graph, let $p : E(G) \to \mathbb{N}$, and let $k \geq 1$ be an integer. Then each of the following statements implies the next:*

(i) *$p(\delta(v)) < k$ for every vertex v, and G has p-carving-width $\geq k$*

(ii) *$p(\delta(v)) < k$ for every vertex v, and G has a tilt of p-order k*

(iii) *G' has a uniform slope of order $k/2$, where G' is obtained from G^* by subdividing e^* $p(e) - 1$ times, for each $e \in E(G)$*

(iv) *G has a connected antipodality of p-range $\geq k$*

(v) *G has an antipodality of p-range $\geq k$.*

In particular, since (i)$\Rightarrow$(v) we see the "only if" part of (4.1) holds.

5. Bond carvings

Now we begin the proof of the "if" part of (4.1); it will be completed in the next section. Let G be a connected graph. A carving $\mathscr{C}$ in $V(G)$ is a *bond* carving if $\delta(X)$ is a bond for all $X \in \mathscr{C}$. The main result of this section is the following.

(5.1) *Let G be a 2-connected graph with $|V(G)| \geq 2$ and with p-carving-width $< k$, where $p: E(G) \to \mathbb{N}$. Then there is a bond carving $\mathscr{C}$ in $V(G)$ such that $p(\delta(X)) < k$ for all $X \in \mathscr{C}$.*

Let us say that $X \subseteq V(G)$ is *connected* if $G \mid X$ is connected. Then a carving $\mathscr{C}$ in $V(G)$ is a bond carving if and only if each $X \in \mathscr{C}$ is connected. We shall need the following easy lemma (implied by (1.1)), the proof of which we omit.

(5.2) *If $\mathscr{C}$ is a carving in a set V then*

(i) *if $X \in \mathscr{C}$ then $V - X \in \mathscr{C}$*

(ii) *if $X \in \mathscr{C}$ and $|X| \geq 2$ then there is a unique choice of Y, $Z \in \mathscr{C}$ such that $Y \cup Z = X$ and $Y \cap Z = \emptyset$.*

If X, $Y \subseteq V(G)$ are disjoint we denote by $\delta(X,Y)$ the set of all edges of G with one end in X and the other in Y. Thus $\delta(X, V(G) - X) = \delta(X)$. Now let G be connected; then if X, Y, $Z \subseteq V(G)$ are mutually disjoint and nonempty, and have union $V(G)$, then at most one of $\delta(X,Y)$, $\delta(Y,Z)$ $\delta(Z,X)$ is empty. If $\delta(X,Y) = \emptyset$ we define $\mu(\{X,Y,Z\}) = |Z| - 1$, and similarly if $\delta(Y,Z) = \emptyset$ or $\delta(Z,X) = \emptyset$. If none of the three is empty we define $\mu(\{X,Y,Z\}) = 0$.

Let $\mathscr{C}$ be a carving in $V(G)$. A *triad* of $\mathscr{C}$ is a set $\{X,Y,Z\}$ of three members of $\mathscr{C}$, mutually disjoint and with union $V(G)$. By (5.2) (ii), every member X of $\mathscr{C}$ is in at most one triad of $\mathscr{C}$, exactly one if and only if $|X| \leq |V(G)| - 2$. We define $\mu(C)$ to be $\Sigma \mu(\{X,Y,Z\})$, the sum being taken over all triads $\{X,Y,Z\}$ of $\mathscr{C}$. We need the following lemma.

(5.3) *Let G be a connected graph with $|V(G)| \geq 2$, and let $\mathscr{C}$ be a carving in $V(G)$. Let A_1, A_2, B_1, $B_2 \in \mathscr{C}$ be mutually disjoint, with union $V(G)$, and with $A_1 \cup A_2 \in \mathscr{C}$. Let $\delta(A_1, B_1) \neq \emptyset \neq \delta(A_2, B_2)$, and let $\delta(A_1, A_2) = \emptyset$. Let $\mathscr{C}'$ be the carving*

$$\mathscr{C}' = (\mathscr{C} - \{A_1 \cup A_2, B_1 \cup B_2\}) \cup \{A_1 \cup B_1, A_2 \cup B_2\}.$$

Then $\mu(\mathscr{C}') < \mu(\mathscr{C})$.

Proof. Clearly $\mathscr{C}'$ is a carving, and

$$\mu(\mathscr{C}) - \mu(\mathscr{C}') = \mu(\{A_1 \cup A_2, B_1, B_2\}) + \mu(\{A_1, A_2, B_1 \cup B_2\}) - \\ - \mu(\{A_1 \cup B_1, A_2, B_2\}) - \mu(\{A_1, B_1, A_2 \cup B_2\}).$$

Moreover, $\mu(\{A_1, A_2, B_1 \cup B_2\}) = |B_1 \cup B_2| - 1$ since $\delta(A_1, A_2) = \emptyset$, and we may therefore assume, for a contradiction, that

(1) $\mu(\{A_1 \cup B_1, A_2, B_2\}) + \mu(\{A_1, B_1, A_2 \cup B_2\}) \geq \mu(\{A_1 \cup A_2, B_1, B_2\}) + |B_1 \cup B_2| - 1.$

In particular, from (1) it follows that

$$\mu(\{A_1 \cup B_1, A_2, B_2\}) - (|B_2| - 1) + \mu(\{A_1, B_1, A_2 \cup B_2\}) - (|B_1| - 1) > 0$$

and so from the symmetry between $A_1 \cup B_1$ and $A_2 \cup B_2$, we may assume without loss of generality that

$$\mu(\{A_1, B_1, A_2 \cup B_2\}) > |B_1| - 1.$$

We deduce that $\delta(A_1, A_2 \cup B_2) \neq \emptyset$; but since $\delta(A_1, B_1) \neq \emptyset$ by hypothesis, it follows that $\mu(\{A_1, B_1, A_2 \cup B_2\}) = |A_1| - 1$ and $\delta(B_1, A_2 \cup B_2) = \emptyset$. Since

$$\delta(B_1, B_2) \subseteq \delta(B_1, A_2 \cup B_2) = \emptyset$$

we deduce that $\mu(\{A_1 \cup A_2, B_1, B_2\}) = |A_1 \cup A_2| - 1$. From (1), we find that

$$\mu(\{A_1 \cup B_1, A_2, B_2\}) + |A_1| - 1 \geq |A_1 \cup A_2| - 1 + |B_1 \cup B_2| - 1,$$

and hence

$$\mu(\{A_1 \cup B_1, A_2, B_2\}) \geq |A_2 \cup B_1 \cup B_2| - 1 > \max(|A_2| - 1, |B_2| - 1).$$

But since $\delta(A_2, B_2) \neq \emptyset$, it follows that $\mu(A_1 \cup B_1, A_2, B_2)$ is one of $|A_2| - 1$, $|B_2| - 1$, 0. This is a contradiction, as required. ∎

Proof of (5.1). Choose a carving $\mathscr{C}$ in $V(G)$ such that
(1) $p(\delta(X)) < k$ *for all* $X \in \mathscr{C}$, *and*
(2) *subject to* (1), $\mu(\mathscr{C})$ *is minimum.*

We claim that $\mathscr{C}$ is a bond carving. Suppose not; then some $X \in \mathscr{C}$ is not connected, and we may choose such an X, minimal. Then $|X| > 1$, and by (5.2) there exist X_1, $X_2 \in \mathscr{C}$ with $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = X$. From the minimality of X it follows that X_1, X_2 are both connected, and hence $\delta(X_1, X_2) = \emptyset$. Now $\{V(G) - X, X_1, X_2\}$ is a triad of $\mathscr{C}$, and we may therefore choose a triad $\{A_1, A_2, B\}$ of $\mathscr{C}$ such that
(3) $\delta(A_1, A_2) = \emptyset$, *and*
(4) *subject to* (3), $|B|$ *is minimum.*

Since $V(G) - B$ is not connected and G is 2-connected, it follows that $|B| \geq 2$. By (5.2), there exist B_1, $B_2 \in \mathscr{C}$ with $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = B$.
(5) *For* $i = 1,2$ *at least one of* $\delta(A_1, B_i)$, $\delta(A_2, B_i)$ *is non-empty.*

This is because $\{B_1, A_1 \cup A_2, B_2\}$ is a triad of $\mathscr{C}$ and $|B_1| < |B|$, and so from (4), $\delta(A_1 \cup A_2, B_2) \neq \emptyset$, and similarly $\delta(A_1 \cup A_2, B_1) \neq \emptyset$.
(6) *If* $\delta(A_1, B_1)$, $\delta(A_2, B_2)$ *are both non-empty then* $p(\delta(A_1 \cup B_1)) \geq k$.

To see this, define $\mathscr{C}'$ as in (5.3). By (5.3), $\mu(\mathscr{C}') < \mu(\mathscr{C})$, and so by (2), $p(\delta(X)) \geq k$ for some $X \in \mathscr{C}'$. Since $\delta(A_1 \cup B_1) = \delta(A_2 \cup B_2)$, the claim follows.

Similarly,
(7) *If* $\delta(A_2, B_2)$, $\delta(A_2, B_1)$ *are both non-empty then* $p(\delta(A_2 \cup B_2)) \geq k$.

Now since G is connected and $\delta(A_1, A_2) = \emptyset$, at least one of $\delta(A_i, B_1)$, $\delta(A_i, B_2)$ is non-empty, for $i = 1,2$. From (5) we may therefore assume that $\delta(A_1, B_1), \delta(A_2, B_2)$ are both non-empty. From (6), $p(\delta(A_1 \cup B_1)) \geq k$. But $p(\delta(B_1)) < k$, and so $\delta(A_1 \cup B_1) \not\subseteq \delta(B_1)$, and hence $\delta(A_1, B_2) \neq \emptyset$. Similarly, since $\delta(A_2 \cup B_2) \not\subseteq \delta(B_1)$, it follows that $\delta(A_2, B_1) \neq \emptyset$. From (7), $p(\delta(A_1 \cup B_2)) \geq k$. Consequently,

$$\begin{aligned} 2k &\leq p(\delta(A_1 \cup B_1)) + p(\delta(A_1 \cup B_2)) \\ &= p(\delta(A_1, B_2)) + p(\delta(A_2, B_1)) + p(\delta(B_1, B_2)) + \\ &\qquad + p(\delta(A_1, B_1)) + p(\delta(A_2, B_2)) + p(\delta(B_1, B_2)) \\ &= p(\delta(B_1)) + p(\delta(B_2)) < 2k \end{aligned}$$

a contradiction. Thus $\mathscr{C}$ is a bond carving, as required. ∎

6. Carvings and antipodalities

The main result of this section is the following.

(6.1) *Let G be a connected planar graph with $|V(G)| \geq 2$, drawn in a sphere Σ, let G^* be a dual graph and let $p: E(G) \to \mathbb{N}$. Let $k \geq 0$ be an integer, and let α be an antipodality in G of p-range $\geq k$. Then G has p-carving-width $\geq k$.*

Proof. Let us say a *limb* of G is a pair (P, v), where $v \in P \subseteq V(G)$, $\delta(P) \subseteq (v)$, and $V(\alpha(e)) \cap P \neq \emptyset$ for some edge e incident with v. (Thus if (P, v) is a limb and $P \neq V(G)$ then v is a cutvertex of G.)

(1) *If (P, v) is a limb then $V(\alpha(e)) \cap (P - \{v\}) \neq \emptyset$ for every edge e incident with v.*

To prove this, let $e_1, \ldots, e_t$ be the edges of G incident with v, in their cyclic order in the drawing (any loops incident with v occur twice in this sequence). We may assume that $V(\alpha(e_1)) \cap P \neq \emptyset$. Suppose that there exists $i \geq 1$, minimum such that $V(\alpha(e_i)) \cap P = \emptyset$. Since $i > 1$ we deduce that $V(\alpha(e_{i-1})) \cap P \neq \emptyset$. Let H be a component of $\alpha(e_{i-1})$ with $V(H) \cap P \neq \emptyset$. Since $v \notin V(H)$ by (A1), and $\delta(P) \subseteq \delta(v)$, and H is connected, it follows that $V(H) \subseteq P$. Let $r \in R(G)$ be incident with e_{i-1} and e_i. Then $V(H) \cap \alpha(r) \neq \emptyset$ by (A2), and by (A2) again, $\alpha(r) \subseteq V(\alpha(e_i))$. Hence

$$\emptyset \neq V(H) \cap \alpha(r) \subseteq P \cap \alpha(r) \subseteq P \cap V(\alpha(e_i)),$$

contrary to the choice of i. Thus there is no such i and so $V(\alpha(e_i)) \cap P \neq \emptyset$ for $1 \leq i \leq t$. But $v \notin V(\alpha(e_i))$ by (A1), and so (1) holds.

Now $(V(G), v)$ is a limb, for any $v \in V(G)$, because v has valency ≥ 1 in G. Hence we may choose a limb (P, v) with P minimal.

(2) *$P - \{v\}$ is connected.*

Suppose not; then there exist P_1, $P_2 \subseteq P$ such that $P_1 \cup P_2 = P$, $P_1 \cap P_2 = \{v\}$, $\delta(P_1 - \{v\}, P_2 - \{v\}) = \emptyset$, and P_1, $P_2 \neq P$. Choose $e \in E(G)$ incident with v, such that $P \cap V(\alpha(e)) \neq \emptyset$. Then one of $P_1 \cap V(\alpha(e))$, $P_2 \cap V(\alpha(e))$ is non-empty and so one of (P_1, v), (P_2, v) is a limb, contrary to the choice of (P, v). This proves (2).

Since (P, v) is a limb it follows that $V(\alpha(e)) \cap (P - \{v\}) \neq \emptyset$ for some edge incident with v, and so $P \neq \{v\}$. Since $\delta(P) \subseteq \delta(v)$ it follows that $\delta(P - \{v\}, \{v\}) \neq \emptyset$. Let B be a maximal 2-connected subgraph of G containing v and a neighbour of v in P. (A single edge and its ends form a 2-connected subgraph.) Then $V(B) \subseteq P$, because $\delta(P) \subseteq \delta(v)$.

(3) *Every neighbour of v in P belongs to $V(B)$.*

To see this, certainly some neighbour $u_1 \in P$ of v belongs to $V(B)$. Let u_2 be another neighbour of v in P. By (2), there is a circuit C of $G | P$ such that v, u_1, $u_2 \in V(C)$ and hence $|V(B \cap C)| \geq 2$. Consequently $B \cup C$ is 2-connected, and so $B \cup C = B$ from the maximality of B. Hence $u_2 \in V(B)$, and so (3) holds.

For $X \subseteq V(B)$, let $\tilde{X}$ be the unique subset of $V(G)$ satisfying $\tilde{X} \cap V(B) = X$ and $\delta(\tilde{X}) = \delta(X, V(B) - X)$. It is easy to see that if $v \notin X$ then $\tilde{X} \subseteq P - \{v\}$. We suppose, for a contradiction, that B has p-carving-width $< k$. Since B is 2-connected, there is by (5.1) a bond carving $\mathscr{C}$ in $V(B)$ such that for all $X \in \mathscr{C}$, $p(\delta(X, V(B) - X)) <$

k. Hence $p(\delta(\tilde{X}))<k$ for all $X\in\mathscr{C}$. Let $\mathscr{C}'\subseteq\mathscr{C}$ be the set of all $X\in\mathscr{C}$ such that $v\notin X$ and $V(\alpha(e))\cap\tilde{X}\neq\emptyset$ for some $e\in\delta(\tilde{X})$.

(4) $\mathscr{C}'\neq\emptyset$.

To see this, let $X=V(B)-\{v\}$; then $X\in\mathscr{C}$, since $\mathscr{C}$ is a carving in $V(B)$ and $|V(B)|\geq 2$. By (3), $\tilde{X}=P-\{v\}$. Choose $e\in E(B)$ incident with v; then $V(\alpha(e))\cap\tilde{X}\neq\emptyset$ by (1), and since $e\in\delta(\tilde{X})$ it follows that $X\in\mathscr{C}'$. Hence $\mathscr{C}'\neq\emptyset$, as required.

Choose $X\in\mathscr{C}'$, minimal.

(5) $|X|\neq 1$.

Suppose that $X=\{u\}$ say. Then $\delta(\tilde{X})\subseteq\delta(u)$, and $V(\alpha(e))\cap\tilde{X}\neq\emptyset$ for some $e\in\delta(\tilde{X})$ because $X\in\mathscr{C}'$, and so $(\tilde{X},u)$ is a limb. But $\tilde{X}\subseteq P-\{v\}$ contrary to the choice of (P,v). This proves (5).

By (5.2) there exist X_1, $X_2\in\mathscr{C}$ with $X_1\cup X_2=X$ and $X_1\cap X_2=\emptyset$. By the minimality of X, neither X_1 nor X_2 belongs to $\mathscr{C}'$.

(6) *$E(\alpha(e))\cap(\delta(\tilde{X}_1)\cup\delta(\tilde{X}_2))=\emptyset$ for all $e\in\delta(\tilde{X}_1)\cup\delta(\tilde{X}_2)$.*

To see this, $e,f\in\delta(\tilde{X}_1)\cup\delta(\tilde{X}_2)$. Then one of $\delta(\tilde{X})$, $\delta(\tilde{X}_1)$, $\delta(\tilde{X}_2)$, say D, contains both e and f. Since $\mathscr{C}$ is a bond carving of B it follows that D is a bond of G, and hence

$$\{g^*:g\in D\}=E(C)$$

for some circuit C of G^*. Since e^*, $f^*\in E(C)$ and C has p-length $p(D)<k$, we deduce from (A3) that $f\notin E(\alpha(e))$. This proves (6).

(7) *$V(\alpha(e))\cap\tilde{X}_1=\emptyset$ for all $e\in\delta(\tilde{X})$.*

To see this, let $\delta(\tilde{X})=\{e_1,\dots,e_t\}$, such that for $1\leq i\leq t$ there is a region r_i of G incident with e_{i-1}, e_i (where e_0 means e_t). Such a numbering is possible since $\delta(\tilde{X})$ is a bond of B and hence of G, and so $\{e^*:e\in\delta(\tilde{X})\}$ is the edge-set of a circuit of G^*. Since $V(B)-X_2$ is connected (because $V(B)-X_2\in\mathscr{C}$) it follows that $\delta(\tilde{X}_1)\cap\delta(\tilde{X})\neq\emptyset$, and so we may choose the numbering so that $e_1\in\delta(\tilde{X}_1)$. Since $X_1\notin\mathscr{C}$ it follows that $V(\alpha(e_1))\cap\tilde{X}_1=\emptyset$. If possible, choose $i\geq 1$ minimum such that $V(\alpha(e_i))\cap\tilde{X}_1\neq\emptyset$. Then $i>1$, and $V(\alpha(e_{i-1}))\cap\tilde{X}_1=\emptyset$. Let H be a component of $\alpha(e_i)$ with $V(H)\cap\tilde{X}_1\neq\emptyset$. Since $e_i\in\delta(\tilde{X})\subseteq\delta(\tilde{X}_1)\cup\delta(\tilde{X}_2)$, it follows from (6) that $E(H)\cap\delta(\tilde{X}_1)=\emptyset$. Since H is connected we deduce that $V(H)\subseteq\tilde{X}_1$. By (A2), $V(H)\cap\alpha(r_i)\neq\emptyset$, and hence $\alpha(r_i)\cap\tilde{X}_1\neq\emptyset$. By (A2), $\alpha(r_i)\subseteq V(\alpha(e_{i-1}))$, and so $V(\alpha(e_{i-1}))\cap\tilde{X}_1\neq\emptyset$, a contradiction. Thus there is no such i, and the result follows.

Similarly, $V(\alpha(e))\cap\tilde{X}_2=\emptyset$ for all $e\in\delta(\tilde{X})$. Since $\tilde{X}_1\cup\tilde{X}_2=\tilde{X}$ it follows that $V(\alpha(e))\cap\tilde{X}=\emptyset$ for all $e\in\delta(\tilde{X})$, a contradiction, since $X\in\mathscr{C}'$.

We deduce that B has p-carving-width $\geq k$; and hence so does G, because B is a subgraph of G. ∎

We deduce:

(6.2) *Let G be a connected planar graph with $|V(G)|\geq 2$, drawn in a sphere, let G^* be a dual graph, let $p:E(G)\to\mathbb{N}$, and let $k\geq 0$ be an integer. Suppose that $p(\delta(v))<k$ for all $v\in V(G)$. Then the following are equivalent:*

(i) *G has p-carving-width $\geq k$*

(ii) *G has a tilt of p-order k*

(iii) *G' has a uniform slope of order $k/2$, where G' is obtained from G^* by subdividing e^* $p(e)-1$ times, for each $e \in E(G)$*

(iv) *G has a connected antipodality of p-range $\geq k$*

(v) *G has an antipodality of p-range $\geq k$.*

Proof. Since $p(\delta(v))<k$ for all $v \in V(G)$, and $V(G) \neq \emptyset$, it follows that $k \geq 1$. Hence (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v) by (4.7). But (v)$\Rightarrow$(i) by (6.1). ∎

Proof of (4.1). We have already proved the "only if" part of (4.1), as a corollary of (4.7). For the "if" half, let G be a connected planar graph with $|V(G) \geq 2$. Since G has p-carving-width ≥ 0, we may assume that $k \geq 1$. If $p(\delta(v)) \geq k$ some $v \in V(G)$ then G has p-carving-width $\geq k$ since $\{v\} \in \mathscr{C}$ for every p-carving $\mathscr{C}$ in $V(G)$, as required. We may assume then that $p(\delta(v))<k$ for all $v \in V(G)$, and that G has an antipodality of p-range $\geq k$. But then by (6.2), G has p-carving-width $\geq k$, as required. ∎

From (4.1), we obtain an algorithm for p-carving-width in planar graphs, as follows.

(6.3) *Algorithm.*
Input: A planar graph G with $|V(G)| \geq 2$, a function $p: E(G) \rightarrow \mathbb{Z}_+$, and an integer $k \geq 1$.
Output: Decides whether G has p-carving-width $\geq k$.
Running time: $\leq O(m^2)$, where $m=|V(G)+|E(G)|$, if arithmetic operations can be performed in unit time.
Description:

Step 1. *Delete every edge e of G with $p(e)=0$, forming G'.*

Such edges have no effect on the p-carving-width.

Step 2. *Compute all the components, $G_1, \ldots, G_t$ say, of G' which have ≥ 2 vertices.*

Step 3. *Test if some vertex v of $G_1 \cup \ldots \cup G_t$ has $p(\delta(v)) \geq k$. If so, output "yes" and stop.*

This is correct because by (4.1), if $v \in V(G_i)$ has $p(\delta(v)) \geq k$ then G_i, and hence G, has p-carving-width $\geq k$.

Step 4. *For $1 \leq i \leq t$, find a drawing of G_i in a sphere, and a dual graph. Test if any of $G_1, \ldots, G_t$ has an antipodality of p-range $\geq k$, using (2.4). If so, output "yes", and otherwise output "no", and stop.*

This is correct because if some G_i has an antipodality of p-range $\geq k$ then it and hence G has p-carving-width $\geq k$ by (4.1). Otherwise, by (4.1), each G_i has p-carving-width $<k$. Since the one-vertex components of G' have no effect on p-carving-width, it follows easily that G' has p-carving-width $<k$, and hence so does G, as required.

To estimate running time, we see that steps 1–3 can be performed in time $O(m)$. If $m_i=|V(G_i)|+|E(G_i)|$, then the application of (2.4) to G_i in step 4 takes time $\leq O(m_i^2)$, and since $\sum m_i^2 \leq m^2$ it follows that the total running time is $\leq O(m^2)$. ∎

7. Branch-width and carving-width

In this section we describe how to compute the branch-width of a planar graph. In fact our method applies equally well to "planar hypergraphs", and since in any case we shall need to discuss hypergraph branch-width in the next section, we have expressed our results in terms of hypergraphs. Thus, a *hypergraph* G consists of a finite set $V(G)$ of *vertices*, a finite set $E(G)$ of *edges*, and an incidence relation between them. The vertices incident with an edge e are called the *ends* of e. For $X \subseteq E(G)$ we define $\partial(X) = \partial_G(X)$ to be the set of all $v \in V(G)$ incident with an edge in X and with an edge in $E(G) - X$; and the *branch-width* of G is the minimum, over all carvings $\mathscr{C}$ in $E(G)$, of the maximum, over all $X \in \mathscr{C}$, of $|\partial(X)|$ (or zero, if $|E(G)| \leq 1$). If the hypergraph G is a graph (that is, if each edge has one or two ends) this coincides with our previous definition.

A hypergraph H is a *subhypergraph* of a hypergraph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and every edge of H has the same ends in H and in G. If H_1, H_2 are subhypergraphs of H, then so are $H_1 \cap H_2$, $H_1 \cup H_2$ defined in the natural way. A *separation* of G is a pair (A, B) of subhypergraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$; and its *order* is $|V(A \cap B)|$. Let $k \geq 1$ be an integer. A *tangle* in G of order k is a set $\mathscr{T}$ of separations of G, such that

(i) for any separation (A, B) of G, $\mathscr{T}$ contains one of (A, B), (B, A) if and only if (A, B) has order $< k$

(ii) if $(A_i, B_i) \in \mathscr{T}$ $(i = 1, 2, 3)$ then $A_1 \cup A_2 \cup A_3 \neq G$

(iii) if $(A, B) \in \mathscr{T}$ then $V(A) \neq V(G)$.

We define $\gamma(G)$ to be the maximum, over all $e \in E(G)$, of the number of ends of e (or $\gamma(G) = 0$ if $E(G) = \emptyset$). We shall need the following [4, theorem (4.3)].

(7.1) *Let G be a hypergraph with $\gamma(G) > 0$ and let $k \geq 1$ be an integer. Then G has a tangle of order k if and only if either $\gamma(G) \geq k$ or G has branch-width $\geq k$.*

Let G be a hypergraph; then $I(G)$, the *incidence* graph of G, is the simple bipartite graph with vertex set $V(G) \cup E(G)$, in which $v \in V(G)$ is adjacent to $e \in E(G)$ if and only if v is an end of e in G. We see that G is determined by $I(G)$; and if G is a graph, then $I(G)$ is obtained from G (up to isomorphism) by replacing every edge of G by two edges in series. Thus, if G is a graph then G is planar if and only if $I(G)$ is planar; and that motivates the definition that a hypergraph G is said to be *planar* if $I(G)$ is planar. (It is easy to see that this coincides with the usual definition of planarity for hypergraphs, where edges are represented by closed discs in a sphere, and their ends by points in the boundaries of the discs.)

Take a drawing of $I(G)$ in a sphere. Let M be a graph with vertex set $E(G)$, and let C_v $(v \in V(G))$ be circuits of M, with the following properties:

(i) the circuits C_v $(v \in V(G))$ are mutually edge-disjoint and have union M

(ii) for each $v \in V(G)$, let the neighbours of v in $I(G)$ be $x_1, \dots, x_t$, enumerated according to the cyclic order of the edges $vx_1, \dots, vx_t$ in the drawing of $I(G)$; then C_v has vertex set $\{x_1, \dots, x_t\}$, and in C_v x_{i-1} is adjacent to x_i $(1 \leq i \leq t)$, where x_0 means x_t.

In these circumstances M is called a *medial graph* of G. A hypergraph G is *connected* if $I(G)$ is connected. It is easy to see that every connected planar hypergraph G with $E(G) \neq \emptyset$ has a medial graph, and every medial graph is planar. We shall show the following.

(7.2) *Let G be a connected planar hypergraph with $|E(G)|\geq 2$, and let M be the medial graph of G. Then the branch-width of G is half the carving-width of M.*

Proof. Let C_v $(v\in V(G))$ be circuits of M as in the definition of "medial graph". Let G have branch-width β, and let M have carving-width κ. We must show that $\beta=\kappa/2$. First, we prove that $\beta\leq\kappa/2$. For let $\mathscr{C}$ be a carving in $V(M)$ such that $|\delta_m(X)|\leq\kappa$ for all $X\in\mathscr{C}$. Since $V(M)=E(G)$ it follows that $\mathscr{C}$ is a carving in $E(G)$. Moreover, for all $X\in\mathscr{C}$,

$$\begin{aligned}|\partial_G(X)| &= |\{v\in V(G): v \text{ is incident in } G \text{ with an edge in } X\\ &\qquad\qquad \text{and an edge in } E(G)-X\}|\\ &= |\{v\in V(G):\ \text{in } I(G), v \text{ has a neighbour in } X\\ &\qquad\qquad \text{and in } E(G)-X\}|\\ &= |\{v\in V(G): X\cap V(C_v)\neq\emptyset \text{ and } (E(G)-X)\cap V(C_v)\neq\emptyset\}|\\ &\leq \sum_{v\in V(G)} |E(C_v)\cap\delta_M(X)|/2 = |\delta_M(X)|/2\leq\kappa/2.\end{aligned}$$

We deduce that $\beta\leq\kappa/2$.

For the reverse inequality, suppose first that some $v\in V(G)$ is an end of exactly one edge e of G. Let G' be the hypergraph with $E(G')=E(G)$ and $V(G')=V(G)-\{v\}$, in which $u\in V(G')$ and $f\in E(G')$ are incident if and only if they are incident in G. It is easy to see that G and G' have the same branch-width. Moreover, C_v is a 1-edge circuit of M, and a medial graph M' for G' can be obtained from M by deleting the loop in $E(C_v)$. Clearly M and M' have the same carving-width, because loops do not affect carving-width. Hence it suffices to show that G' has branch-width half the carving-width of M'. By repeating this process, it follows that we may assume that

(1) *There is no $v\in V(G)$ incident with exactly one edge of G.*

We may also assume that $\kappa\geq 1$, for otherwise $\beta\geq\kappa/2$ as required. Consequently $E(M)\neq\emptyset$. We claim that we may assume that

(2) *Each edge of G has $<\kappa/2$ ends in G.*

Suppose that $e\in E(G)$ has $\geq\kappa/2$ ends in G. Now $\{e\}\in\mathscr{C}$ for every carving $\mathscr{C}$ in $E(G)$. Since $|\partial_G(\{e\})|\geq\kappa/2$ by (1), we deduce that $\beta\geq\kappa/2$ as required. Hence we may assume that (2) holds.

Now M has carving-width κ and is planar and connected and $E(M)\neq\emptyset$, and so from (4.3) and (4.4), either $|\delta_M(e)|\geq\kappa$ for some $e\in V(M)$, or M^* has a uniform slope of order $\kappa/2$, where M^* is a dual graph of M. In the first case, since $M=\bigcup(C_v: v\in V(G))$ and each $E(C_v)$ includes at most two edges in $\delta_M(e)$, it follows that $e\in V(C_v)$ for at least $\kappa/2$ values of $v\in V(G)$; that is, e has $\geq\kappa/2$ ends in G, contrary to (2). We deduce that M^* has a uniform slope of order $\kappa/2$. Consequently, by [5,theorems (6.1) and (6.5)], G has a tangle of order $\geq\kappa/2$ (for in the terminology of [5], M^* is the graph of a "radial drawing" of G). If $\gamma(G)=0$ then because G is connected and $E(G)\neq\emptyset$, it follows that $|E(G)|=1$ and $V(G)=\emptyset$, and so M has carving-width 0, a contradiction. Thus $\gamma(G)>0$. By (7.1) and (2), G has branch-width $\geq\kappa/2$, as required. ∎

(7.2) yields the following.

(7.3) *Algorithm.*
Input: A planar hypergraph G, and an integer $k \geq 1$.
Output: Decides whether G has branch-width $\geq k$.
Running time: $O(m^2)$ where $m = |V(G)| + |E(G)|$.
Description:

Step 1. *Compute $I(G)$ and find its components, $H_1, \ldots, H_t$ say.*

There correspond connected subhypergraphs $G_1, \ldots, G_t$ of G with union G, with $V(G_i \cap G_j) = \emptyset = E(G_i \cap G_j)$ $(i \neq j)$, and where $H_i = I(G)$ $(1 \leq i \leq t)$.

Step 2. *For $1 \leq i \leq t$, if $V(H_i)$ contains at least two members of $E(G)$, find a drawing of H_i in a sphere, and compute the corresponding medial graph M_i of G_i.*

Step 3. *For each medial graph M_i, test if M_i has carving-width $\geq 2k$. If some M_i has carving-width $\geq 2k$, output "yes", and otherwise output "no"; and stop.*

This is correct because if some M_i has carving-width $\geq 2k$, then by (7.2) G_i has branch-width $\geq k$, and hence so does G. If no M_i has carving-width $\geq 2k$, then each corresponding G_i has branch-width $< k$ by (7.2). Since every other G_i has at most one edge, it follows that $G_1, \ldots, G_t$ all have branch-width $< k$, and hence so does G.

Since (summing over all i with $|V(H_i) \cap E(G)| \geq 2$)

$$\sum_i (|E(M_i)| + |V(M_i)|)^2 \leq (\sum_i |E(M_i)| + |V(M_i)|)^2$$
$$\leq (\sum_i |E(H_i)| + |E(G_i)|)^2 \leq (|E(I(G))| + |E(G)|)^2$$

and since $|E(I(G))| \leq 2(|V(G)| + |E(G)|) - 3$ because $I(G)$ is planar and simple and has $|V(G))| + |E(G)|$ vertices) we deduce that the algorithm has running time $\leq O(m^2)$, as claimed. ∎

8. Some NP-completeness results

We have seen that for planar graphs and hypergraphs, one can compute branch-width and carving-width in polynomial time. In this section we show that for general graphs both problems are NP-hard. We begin with the following result of Garey, Johnson and Stockmeyer [3].

(8.1) *The following problem is NP-complete.*
Instance: A graph G, two vertices s, t of G, and an integer $k \geq 0$.
Question: Is there a partition (A, B) of $V(G)$ with $|A| = |B|$, $s \in A$, $t \in B$ and $|\delta(A)| \leq k$?

We deduce

(8.2) *The following problem is NP-complete.*
Instance: A graph G, and an integer $k \geq 0$.
Question: is there a partition (A,B) of $V(G)$ with $|A|=|B|$ and $|\delta(A)| \leq k$?

Proof. We shall reduce the problem of (8.1) to that of (8.2). For let G, s, t, k be as in (8.1). Let $|V(G)|=2n$. We may assume that n is an integer. Let G' be obtained from G by adding $k+1$ parallel edges joining u, v for every unordered pair $\{u,v\} \neq \{s,t\}$ of distinct vertices. Let $k'=n^2(k+1)-1$. Let (A,B) be a partition of $V(G)$ with $|A|=|B|$. We claim that
(1) *$|\delta_{G'}(A)| \leq k'$ if and only if A contains exactly one of s, t and $|\delta_G(A)| \leq k$.*

To show this, there are two cases. If A contains exactly one of s, t then

$$|\delta_{G'}(A)| = (n^2-1)(k+1) + |\delta_G(a)|$$

and so $|\delta_{G'}(A)| \leq k'$ if and only if $|\delta_G(A)| \leq k$. If A contains both or neither of s, t, then

$$|\delta_{G'}(A)| = n^2(k+1) + |\delta_G(A)| > k'.$$

The claim follows.

From (1) we see that the problem of (8.1) is polynomially reducible to that of (8.2), and so the result follows from (8.1). ∎

Actually, the proof given in [3] of (8.1) can also serve as a proof of (8.2).

(8.3) *The following problem is NP-complete.*
Instance: A graph G, and an integer $k \geq 0$.
Question: Is there a partition (A,B,C) of $V(G)$ with $|A|=|B|=|C|$ such that $|\delta(A)|$, $|\delta(B)|$, $|\delta(C)| \leq k$?

Proof. We shall reduce (8.2) to (8.3). Let G, k be as in (8.2), with $|V(G)|=2n$. We may assume that n is an integer. Let H be a graph obtained from a complete graph K_n by replacing each edge by $k+1$ parallel edges. Let G' be the disjoint union of G and H.
(1) *There is a partition (A,B) of $V(G)$ with $|A|=|B|$ and $|\delta_G(A)| \leq k$, if and only if there is a partition (A,B,C) of $V(G')$ with $|A|=|B|=|C|$ and $|\delta_{G'}(A)|$, $|\delta_{G'}(B)|$, $|\delta_{G'}(C)| \leq k$.*

Certainly, given (A,B), we may take $C=V(H)$. Conversely, given A, B, C, since $|\delta_{G'}(A)|$, $|\delta_{G'}(B)|$, $|\delta_{G'}(C)| \leq k$ it follows that no two vertices of H belong to distinct members of $\{A,B,C\}$; that is, we may assume that $V(H) \subseteq C$. Since $|V(H)|=n=|C|$ we deduce that $V(H)=C$, and so $|\delta_G(A)| \leq k$. This proves (1).

The result follows from (1) and (8.3). ∎

(8.4) *The following problem is NP-complete.*
Instance: A graph G with $|V(G)| \geq 2$, and an integer $k \geq 0$.
Question Does G have carving-width $\leq k$?

Proof. We shall reduce (8.3) to (8.4). Let G, k be as in (8.3), and let $|V(G)| = 3n$. We may assume that n is an integer, and $n \geq 1$. Also we may assume that $k < |E(G)|$ (for otherwise the partition exists). Let $m=|E(G)|$, and let G' be obtained from G by adding m parallel edges joining every pair of distinct vertices. Hence
(1) *For $X \subseteq V(G)$, $|\delta_{G'}(X)|=|\delta_G(X)|+m|X|(3n-|X|)$.*

Let $k' = 2mn^2 + k$.

(2) *If there is a partition* (A,B,C) *of* $V(G)$ *with* $|A| = |B| = |C|$ *such that* $|\delta_G(A)|$, $|\delta_G(B)|$, $|\delta_G(C)| \le k$, *then* G' *has carving-width* $\le k'$.

To see this, let $\mathscr{C}_0 = \{A,B,C,A\cup B,A\cup C,B\cup C\}$, and let $\mathscr{C}$ be a carving in $V(G)$ with $\mathscr{C}_0 \subseteq \mathscr{C}$. Now for each $X \in \mathscr{C}$, either X is a subset of one of A, B, C or X is a superset of one of $A\cup B$, $A\cup C$, $B\cup C$, and so in either case $|X|\,(3n - |X|) \le 2n^2$, with strict inequality unless $X \in \mathscr{C}_0$. This if $X \in \mathscr{C} - \mathscr{C}_0$ then by (1),

$$|\delta_{G'}(X)| = |\delta_G(X)| + m|X|\,(3n - |X|) \le \delta_G(X) + m(2n^2 - 1) \le 2mn^2 \le k'$$

because $|\delta_G(X)| \le m$. Since $|\delta_G(X)| \le k$ and hence $|\delta_{G'}(X)| \le k'$ for all $X \in \mathscr{C}_0$, we deduce that $|\delta_{G'}(X) \le k'$ for all $X \in \mathscr{C}$, as required.

(3) *If* G' *has carving-width* $\le k'$ *then there is a partition* (A,B,C) *of* $V(G)$ *with* $|A| = |B| = |C|$ *such that* $|\delta_G(A)|$, $|\delta_G(B)|$, $|\delta_G(C)| \le k$.

To show this, let $\mathscr{C}$ be a carving in $V(G')$ such that $|\delta_{G'}(X)| \le k'$ for all $X \in \mathscr{C}$. Since $n \ge 1$ and hence $C \ne \emptyset$, there exists $A \in \mathscr{C}$ with $|A| < 2n$, by (5.2)(i). Choose $A \in \mathscr{C}$ with $|A| < 2n$ such that $|A|$ is maximum. From (1), we deduce that

$$|\delta_G(A)| + m|A|\,(3n - |A|) = |\delta_{G'}(A)| \le k' = 2mn^2 + k < m(2n^2 + 1)$$

and so $|A|\ (3n - |A|) < 2n^2 + 1$. Since $|A| < 2n$, we deduce that $|A| \le n$. Thus $|V(G) - A| \ge 2n \ge 2$, and so by (5.2)(ii), there exist B, $C \in \mathscr{C}$ with $B \cap C = \emptyset$ and $B \cup C = V(G) - A$. Now $A \cup B \in \mathscr{C}$ and $|A\cup B| > |A|$, and so $|A \cup B| \ge 2n$ from the choice of A. Consequently $|C| = 3n - |A\cup B| \le n$, and similarly $|B| \le n$. Since $|A| \le n$ and $|A\cup B\cup C| = 3n$, it follows that $|A| = |B| = |C| = n$. But as we showed above,

$$|\delta_G(A)| + m|A|\,(3n - |A|) \le 2mn^2 + k,$$

and since $|A| = n$, we deduce that $|\delta_G(A)| \le k$; and similarly $|\delta_G(B)|$, $|\delta_G(C)| \le k$. This proves (3).

From (2), (3) and (8.3), the results follows. ∎

We remark that in (8.4) G is not constrained to be simple; but even if G is constrained to be simple the problem is still NP-complete. To see this, let G, k be an input to (8.4); we may assume that $k \ge 2$ and G is loopless. Let G' be obtained from G by subdividing each edge once. It easy to see that G has carving-width $\le k$ if and only if G' does.

Now we turn to branch-width. From (8.4) we have immediately the following (for given G as in (8.4), let H be the hypergraph with $E(H) = V(G)$ and $V(H) = E(G)$, in which $v \in V(G)$, $e \in E(G)$ are incident if and only if they are incident in G; then the branch-width of H equals the carving-width of G).

(8.5) *The following problem is NP-complete.*
Instance: A hypergraph G*, and an integer* $k \ge 0$.
Question: Does G *have branch-width* $\le k$?

We would like to prove (8.5) for graph instead of hypergraphs. Our method is, given a hypergraph G, to construct a graph with the same branch-width. One might try replacing each edge of G by a complete graph, but that does not work. (For instance, if $|V(G)| = n$ and $|E(G)| = 2$, and both edges are incident with every vertex, then G has branch-width n. But if we replace both edges by cliques we obtain a graph with branch-width about $\frac{2}{3}n$.) Instead, we shall replace each edge of G, with ends $u_1,\ldots,u_t$ say, by a complete bipartite graph with vertex set $\{u_1,\ldots,u_t,v_1,\ldots,v_t\}$ (where $v_1,\ldots,v_t$ are new vertices) in which each u_i is adjacent to each v_j. We shall see that this produces a graph with the same branch-width as G except in trivial cases. To show this, it is convenient to replace one edge of G at a time; and instead of working directly with branch-width, we use tangles and (7.1).

Let us say a separation (C,D) of a hypergraph is *special* if $|V(C)|=2|V(C\cap D)|$, and C is a (simple) complete bipartite graph with bipartition $(V(C\cap D),\ V(C)-V(D))$.

A separation (C,D) of a hypergraph G is *titanic* for every triple (X,Y,Z) of subhypergraphs of C such that $X\cup Y\cup Z=C$ and $E(X)$, $E(Y)$, $E(Z)$ are mutually disjoint, at least one of the following so-called "titanic inequalities" holds:

$$|V((X\cup Y)\cap Z)|\geq|V((X\cup Y)\cap D)|$$
$$|V((Y\cup Z)\cap X)|\geq|V((Y\cup Z)\cap D)|$$
$$|V((Z\cup X)\cap Y)|\geq|V((Z\cup X)\cap D)|.$$

We need the following lemma.

(8.6) *Let (C,D) be a separation of a hypergraph G. If either*
(i) $|E(C)|=1$, $E(C)=\{e\}$ *say, and $V(C)$ is the set of ends of e, or*
(ii) (C,D) *is special*
then (C,D) is titanic.

Proof. Let X, Y, Z be as in the definition of "titanic". If (i) holds, we may assume $e\in E(X)$, and hence every end of e is a vertex of X. Hence $V(C)\subseteq V(X)$, and so $X=C$. But then

$$|V((Y\cup Z)\cap X)|=|V(Y\cup Z)|\geq|V((Y\cup Z)\cap D)|$$

and so one of the titanic inequalities holds, as required. We assume then that (ii) holds.
(1) *We may assume that $V(C)-V(D)\subseteq V(Y\cup Z)$.*

To see this, suppose that $a\in V(C)-V(D)$ and $a\notin V(Y\cup Z)$. Then $a\in V(X)$. Since for all $b\in V(C\cap D)$ there is an edge of C with ends a, b, and this edge does not belong to Y or Z since $a\notin V(Y)$, $V(Z)$, it follows that this edge belongs to X and in particular $b\in V(X)$. Hence $V(C\cap D)\subseteq V(X)$, and so

$$|V((Y\cup Z)\cap X)|\geq|V((Y\cup Z)\cap C\cap D)|=|V((Y\cup Z)\cap D)|;$$

and a titanic inequality holds, as required.
(2) *We may assume that $V(Y\cup Z)=V(C)$.*

To see this, suppose that $b\in V(C\cap D)$ and $b\notin V(Y\cup Z)$. Then every edge of C incident with b belongs to X, and so $V(C)-V(D)\subseteq V(X)$. But by (1), $V(C)-V(D)\subseteq V(Y\cup Z)$, and so

$$|V((Y\cup Z)\cap X)\geq|V(C)-V(D)|=|V(C\cap D)|\geq|V((Y\cup Z)\cap D)|;$$

and a titanic inequality holds, as required. Thus we may assume that $V(C\cap D)\subseteq V(Y\cup Z)$, and hence $V(Y\cup Z)=V(C)$ from (1).

Similarly we may assume that $V(X\cup Y)=V(X\cup Z)=V(C)$. Hence we may assume that $|V(X)|\geq 1/2|V(C)|$. But $V(Y\cup Z)=V(C)$, and so

$$|V((Y\cup Z)\cap X)| = |V(X) \geq 1/2|V(C)| = |V(C\cap D)| \geq |V((Y\cup Z)\cap D)|$$

as required. ∎

We shall also need the following lemma.

(8.7) *Let (C,D) be a special separation of a hypergraph G, of order t. Let $\mathcal{T}$ be a tangle in G of order $k>\max(t,2)$. Then $(C,D)\in\mathcal{T}$.*

Proof. Suppose that $(C,D)\notin\mathcal{T}$. Since (C,D) has order $t<k$ it follows that $(D,C)\in\mathcal{T}$. Choose $(A,B)\in\mathcal{T}$ with $D\subseteq A$ and $B\subseteq C$, such that $|V(A)|+|E(A)|-|V(B)|-|E(B)|$ is maximum. Evidently for all $(A',B')\in\mathcal{T}$, if $A\subseteq A'$ and $B'\subseteq B$ then $(A',B')=(A,B)$. Since $B\subseteq C$ it follows that B is a graph. By [4, theorem (2.8)], we deduce

(1) *(A,B) has order $k-1$, and for any separation (B_1,B_2) of B such that $B_1,B_2\neq B$,*

$$|V(B_1\cap B_2)| > \min(|V(A\cap B_1),|V(A\cap B_2)|).$$

In particular, B is connected and every edge of B has an end in $V(B)-V(A)$.

Consequently there is no separation (B_1,B_2) of B with B_1, $B_2\neq B$ and such that $B_1\cap B_2\subseteq A$; and so $B\backslash V(A\cap B)$ is connected. Since $V(D)$ meets every edge of C and hence $V(A)$ meets every edge of B, it follows that $|V(B)-V(A)|\leq 1$. But $V(A)\neq V(G)$ since $(A,B)\in\mathcal{T}$, and so $V(B)-V(A)=\{v\}$, for some vertex v. Then $v\in V(C)-V(D)$, since $V(B)-V(A)\subseteq V(C)-V(D)$. By (1), $V(B)\subseteq V(C\cap D)\cup\{v\}$. By (1) again, $|V(B)\cap V(C\cap D)|\leq 1$, and so (A,B) has order ≤ 1, By (1), $k-1\leq 1$, a contradiction. ∎

The following is [4, theorem (8.3)].

(8.8) *Let (C,D) be a separation of a hypergraph G, and let (C',D) be a separation of a hypergraph G', with $C\cap D=C'\cap D$. Let $\mathcal{T}$ be a tangle in G of order $k\geq 2$ with $(C,D)\in\mathcal{T}$, and let (C',D) be titanic. Let $\mathcal{T}'$ be the set of all separations (A',B') of G' of order $<k$ such that there exists $(A,B)\in\mathcal{T}$ with $E(A\cap D)=E(A'\cap D)$. Then $\mathcal{T}'$ is a tangle in G' of order k.*

Let e be an edge of a hypergraph G, and let $G\backslash e$ denote the subhypergraph with vertex set $V(G)$ and edge set $E(G)-\{e\}$. Let G' be a hypergraph with $G\backslash e\subseteq G'$, and let $(C,G\backslash e)$ be a special separation of G'. In these circumstances we say that G' is obtained from G by *expanding* e.

(8.9) *Let G' be obtained from a hypergraph G by expanding an edge e, and let $k>\max(2,\gamma(G))$ be an integer. Then G has branch-width $<k$ if and only if G' has branch-width $<k$.*

Proof. If $\gamma(G)=0$ then $\gamma(G')=0$, and G, G' both have branch-width 0. Thus we may assume that $\gamma(G)>0$, and hence that $\gamma(G')>0$. Let e have t ends. Let C be the hypergraph formed by e and its ends, let $D=G\backslash e$, and let (C',D) be the special separation of G'. Both (C,D) and (C',D) are titanic by (8.6), and both have order t.

Suppose that G has branch-width $\geq k$. Since $\gamma(G) > 0$, it follows from (7.1) that G has a tangle $\mathcal{T}$ of order k. Since $(D,C) \notin \mathcal{T}$ (because $V(D) = V(G)$) and (C,D) has order $t<k$, it follows that $(C,D) \in \mathcal{T}$. By (8.8) we deduce that G' has a tangle of order k. Since $\gamma(G') \leq \max(2, \gamma(G)) < k$, it follows from (7.1) that G' has branch-width $\geq k$, as required.

For the converse, suppose that G' has branch-width $\geq k$. Since $\gamma(G') > 0$, it follows from (7.1) that G' has a tangle $\mathcal{T}'$ of order k. By (8.7), $(C',D) \in \mathcal{T}$. By (8.8), G has a tangle of order k. Since $\gamma(G) < k$ it follows from (7.1) that G has branch-width $\geq k$, as required. ∎

We deduce

(8.10) *The following problem is NP-complete.*
Instance: A simple graph G, and an integer $k \geq 0$.
Question: Does G have branch-width $\leq k$?

Proof. We shall reduce the problem of (8.5) to that of (8.10). Let G, k be as in (8.5). We may assume that $k \geq 2$ (for it is easy to test if G has branch-width ≤ 0 or ≤ 1). Suppose that $v \in V(G)$ is an end of exactly one edge $e \in E(G)$. It is easy to see that G and G' have the same branch-width, where $V(G') = V(G) - \{v\}$, $E(G') = E(G)$, and each $f \in E(G')$ is incident with $u \in V(G')$ if and only if they are incident in G. By repeating this process we reduce the problem of testing if G has branch-width $\leq k$ to testing if some hypergraph H has branch-width $\leq k$, in which no vertex is incident with just one edge. In other words, we may assume that no vertex of G is incident with just one edge of G.

We may assume that $\gamma(G) \leq k$, for if $\gamma(G) > k$ then G does not have branch-width $\leq k$. Now let G' be obtained from G by expanding each edge in turn. Then G' is a simple graph, and it has branch-width $\leq k$ if and only if G has branch-width $\leq k$, by (8.9). The result follows. ∎

9. Remarks

We have seen that we can test in polynomial time if a planar graph has carving-width $\leq k$, but our algorithm does not find the corresponding carving if it exists. This can be overcome at some addition cost in running time, as follows.

(9.1) *Algorithm*
Input: A planar graph G, a function $p: E(G) \to \mathbb{Z}_+$, and an integer $k \geq 0$.
Output: A carving $\mathscr{C}$ in $V(G)$ such that $p(\delta(X)) < k$ for all $X \in \mathscr{C}$, if such a carving exists.
Running time: $O(m^4)$, where $m = |V(G)| + |E(G)|$, if arithmetic operations can be performed in unit time.

Description: If we find a carving for each block of G, it is easy to assemble them using (1.1) to find the desired carving in G. Thus, we may assume that G is 2-connected and loopless. We check, using (6.3), whether G has p-carving-width $< k$, and we may assume the answer is yes. By (5.1) there is a bond carving $\mathscr{C}$ in $V(G)$ such that $p(\delta(X)) < k$ for all $X \in \mathscr{C}$. Since we may assume that $|V(G)| \geq 3$ and hence there exist distinct u, $v \in V(G)$ with $\{u,v\} \in \mathscr{C}$, it follows that

(1) *There exist distinct* u, $v \in V(G)$, *such that*

(i) u, v *are adjacent in* G

(ii) $G\backslash\{u,v\}$ *is connected, and*

(iii) *the graph* G' *obtained from* G *by contracting all edges with ends* $\{u,v\}$ *has* p*-carving-width* $<k$.

Hence we may find such a pair u, v, by testing all adjacent pairs u, v to see if (ii) and (iii) above are satisfied (we test (iii) using (6.3)).

But given u, v as in (1), and given a carving $\mathscr{C}'$ in $V(G')$ such that $p(\delta_{G'}(X) < k$ for all $X \in \mathscr{C}'$, it is easy to construct the desired carving in $V(G)$ (for $p(\delta(u))$, $p(\delta(v)) < k$ since G has p-carving-width $<k$). Thus it suffices to find the carving in $V(G')$. But G' is loopless and 2-connected (because $G\backslash\{u,v\}$ is connected) and so we may continue the process.

The algorithm then, for a loopless 2-connected graph G, is as follows. Set $G_1 = G$. Iteratively, for $1 \leq i \leq |V(G)|-2$, we find u_i, $v_i \in V(G_i)$ as in (1), and let G_{i+1} be obtained from G_i by contracting all edges between u_i and v_i. Now we find a bond carving $\mathscr{C}_i$ for $V(G_i)$ such that $p(\delta(X)) < k$ for all $X \in \mathscr{C}_i$, for $i = |V(G)|-1$, $V(G)|-2, \ldots, 1$ in turn; and then $\mathscr{C}_1$ is the required output. ∎

Secondly, it is natural to ask, what about computing the cut-width of a planar graph? A graph G has *cut-width* $\leq k$ if there is an ordering $v_1, \ldots, v_n$ of $V(G)$ such that for $1 \leq i \leq n-1$,

$$|\delta(\{v_1, \ldots, v_i\})| \leq k.$$

For trees, cut-width is computable in polynomial time [6], and for general graphs it is NP-complete [3], but for planar graphs it remains open. It is tempting to try and adapt the methods of the present paper to compute cut-width for planar graphs, but there are difficulties. One is that there appears to be no analogue of (5.1), and another is that we have been unable to formulate an analogue of 'antipodality" so that there is an appropriate version of (4.1). Nevertheless, there is an analogue of (4.3) and (4.4) (see [1] for related material), and so there may be some hope. Our feeling, however, is that computing cut-width is probably NP-hard for planar graphs.

Lastly, an open problem. In practice, it seems that if a planar graph has an antipodality of range $\geq k$ then one can find a drawing of it on a sphere $\{(x,y,z) : x^2+y^2+z^2=1\}$ and an antipodality α of range $\geq k$ such that $\alpha(e)$ tends to be opposite e in the drawing, that is, close to the reflection through the origin of the line segment representing e. Does this have any theoretical basis?

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